STAR PRODUCTS ON COADJOINT ORBITS

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We study properties of a family of algebraic star products defined on coadjoint orbits of semisimple Lie groups. We connect this description with the point of view of differentiable deformations and geometric quantization.

1. FAMILY OF DEFORMATIONS OF THE POLYNOMIALS ON THE ORBIT

Let G be a Lie group of dimension n and $\mathcal{G}$ its Lie algebra. The Kirillov Poisson structure on the dual space $\mathcal{G}^\star$ is given by

$$\{f_1, f_2\}(\lambda) = \langle[(df_1)_\lambda, (df_2)_\lambda], \lambda\rangle, \qquad f_1, f_2 \in C^\infty(\mathcal{G}^*), \quad \lambda \in \mathcal{G}^*. \tag{1}$$

The symplectic leaves of this Poisson structure coincide with the orbits of the coadjoint action of G in $\mathcal{G}$,

$$\langle \mathrm{Ad}^*(\mathrm{g})\lambda, \mathrm{Y}\rangle = \langle \lambda, \mathrm{Ad}(\mathrm{g}^{-1})\mathrm{Y}\rangle \quad \forall\, \mathrm{g} \in \mathrm{G}, \quad \lambda \in \mathcal{G}*, \quad \mathrm{Y} \in \mathcal{G}.$$

Let G be a compact semisimple group of rank n. Then the coadjoint orbits are algebraic varieties. Let $\{p_i\}_{i=1}^m$ be a set of generators of the algebra of G-invariant polynomials on $\mathcal{G}^*$. The coadjoint orbits are determined by the values of these polynomials, that is by the equations

$$p_i = c_i, \qquad i = 1, \ldots m. \tag{2}$$

The regular orbits are those for which the differentials dp_i are independent [1]. They are algebraic symplectic manifolds of dimension $n - m$. The ideal of polynomials vanishing on a

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regular orbit Θ is a prime ideal generated by the relations (2), we will denote it by $\mathcal{I}_0$. The algebra of polynomials on Θ,

$$\mathrm{Pol}(\Theta) = \mathrm{Pol}(\mathcal{G}^*)/\mathcal{I}_0$$

is a Poisson algebra.

A formal deformation of a Poisson algebra $\mathcal{A}$ over $\mathbb{C}$ is a $\mathbb{C}[[h]]$-module $\mathcal{A}_h$ which is isomorphic as a module to $\mathcal{A}[[h]]$, the isomorphism $\Psi : \mathcal{A}[[h]] \to \mathcal{A}_h$ satisfying the conditions:
a. $\psi^{-1}(F_1F_2) = f_1f_2 \mod(h)$ where $F_i \in \mathcal{A}_{[h]}$ are such that $\psi^{-1}(F_i) = f_i \mod(h)$, $f_i \in \mathcal{A}$.
b. $\psi^{-1}(F_1F_2 - F_2F_1) = h\{f_1, f_2\} \mod(h^2)$.

In this definition one can substitute $\mathbb{C}[[h]]$ with $\mathbb{C}[h]$. We will say then that we have a $\mathbb{C}[h]$-deformation. It is clear that a $\mathbb{C}[h]$-deformation can be extended to a formal deformation, while the opposite is not in general true.

Given a deformation $\mathcal{A}_h$, one can make the pull back the product in $\mathcal{A}_h$ to $\mathcal{A}[[h]]$ by the isomorphism Ψ. The product defined in this way is called a star product and is in general given by a formal series

$$f \star g = \Psi^{-1}(\Psi(f)\Psi(g)) = fg + \sum_{n>1} h^n B_n(f,g)$$

where B_n are bilinear operators. If $\mathcal{A}$ is some space of functions and B_n are bidifferential operators we say that the star product is differential. It follows that the star product can be extended to the whole space of C^∞ functions, but only as a formal deformation [4]. By choosing another isomorphism Ψ', one could obtain a star product that is not differential. So a star product that is not differential can be isomorphic to a star product that is differential. We will see examples of this situation later.

A formal (and $\mathbb{C}[h]$) deformation of $\mathrm{Pol}(\mathcal{G}^*)$ is given by the enveloping algebra U_h of the Lie algebra with the bracket $h[\,\cdot\,,\,\cdot\,]$, where $[\,\cdot\,,\,\cdot\,]$ is the bracket on $\mathcal{G}$. (The tensor algebra needs to be taken over $\mathbb{C}[[h]]$). One choice for Ψ is the Weyl map or symmetrizer. If $x_1, \dots, x_n$ are coordinates on $\mathcal{G}^*$ and $X_1, \dots, X_n$ are the corresponding generators of U_h, the Weyl map is

$$W(x_{i_1} \cdots x_{i_p}) = \frac{1}{p!} \sum_{s \in S_p} X_{i_{s(1)}} \cdots X_{i_{s(p)}}.$$

The star product

$$f \star_S g = W^{-1}(W(f)W(g))$$

can be expressed in terms of bidifferential operators, so it can be extended to the whole $C^\infty(\mathcal{G}^*)$.

$\star_S$ is not tangential to the orbits, so it cannot be restricted to one of them. Nevertheless, the formal deformation $U_{[h]}$ can be used to induce a deformation of $\mathrm{Pol}(\Theta)$. This was developed in [2]. The idea is to find an ideal $\mathcal{I}_h$ such that the diagram

$$\begin{array}{ccc} \mathrm{Pol}(\mathcal{G}^*) & \longrightarrow & U_{[h]} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Pol}(\Theta) & \longrightarrow & U_{[h]}/\mathcal{I}_{[h]} \end{array}$$

commutes. The vertical arrows are the natural projections, the horizontal ones indicate deformations. The ideal $\mathcal{I}_h$ is generated by

$$W(p_i) - c_i(h) = P_i - c_i(h), \quad c_i(0) = c_i^0, \qquad i = 1, \dots n.$$

The ideal is $\mathrm{Ad_G}$-invariant since P_i are Casimirs of $U_{[h]}$, so there is a natural action of G on $U_{[h]}/\mathcal{I}_{[h]}$. The same construction works with $\mathbb{C}[h]$. We will consider only $c^i(h)$ such that its degree in h is not bigger than the degree of p_i. In this context one can show that $\mathcal{I}_h$ is a prime ideal [3]. Also, the algebras can be specialized to a value of h, say h_0, by quotienting with the proper ideal generated by $h - h_0$. Analyzing the representations of the specialized algebras, one can see that in general, they are not isomorphic for ideals with $c_i(h) \neq c_i'(h)$.

2. STAR PRODUCTS ON THE POLYNOMIALS ON THE ORBIT

We consider the example of S^2 for clarity, although the argument can be extended to all other compact orbits [2, 3]. $\mathcal{G} = \mathrm{su}(2)$ has dimension 3 with basis $\{X, Y, Z\}$,

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

The unique invariant polynomial is $p(x, y, z) = x^2 + y^2 + z^2$ and the Casimir element is $P = X^2 + Y^2 + Z^2$. The regular orbits are given by

$$x^2 + y^2 + z^2 = c$$

for $c > 0$. A basis of $\mathrm{Pol(S^2)}$ is $\{[x^m y^n z^\nu],\ m, n \in \mathbb{N},\ \nu = 0, 1\}$. An isomorphism $\mathrm{Pol(S^2)[h]} \approx \mathrm{U_h}/\mathcal{I}_\mathrm{h}$, is given by

$$\tilde{\Psi}([x^m y^n z^\nu]) = [X^m Y^n Z^\nu],$$

since $\{[X^mY^nZ^\nu],\ m,n\in\mathbb{N},\ \nu=0,1\}$ is a basis of $U_h/\mathcal{I}_h$. Define the isomorphism $\Psi:\mathrm{Pol(su(2)^*)[h]}\rightarrow \mathrm{U_h}$

$$\Psi(x^my^nz^\nu)=X^mY^nZ^\nu, \qquad m,n\in\mathbb{N}$$
$$\Psi(x^my^nz^r(p-c^0))=X^mY^nZ^r(P-c(h)), \qquad m,n\in\mathbb{N};$$

which sends the ideal $\mathcal{I}_0$ into the ideal $\mathcal{I}_h$, so it passes to the quotient, where it gives the isomorphism $\tilde{\Psi}$. The corresponding star product on $\mathrm{Pol(su(2)^*)[h]}$ restricts to $\mathrm{Pol(S^2)}$.

This star product is not differential, as it is shown in [3], but it is isomorphic to $\star_S$. In addition, for an orbit in a neighborhood of this one, $p-c^0-\Delta c^0=0$, Ψ doesn't preserve the ideal.

Another way of giving a basis is using the decomposition

$$\mathrm{Pol}(\mathcal{G}^*)\approx \mathrm{I}\otimes\mathrm{H},$$

where I is the algebra of invariant polynomials and H is the space of harmonic polynomials, $H\approx \mathrm{Pol}(\Theta)$. We define the isomorphism $\Phi:\mathrm{Pol(su(2)^*)[h]}\rightarrow \mathrm{U_h}$

$$\Phi((p-c)^m\otimes\eta_m)=(P-c(h))^m\tilde{\Phi}(\eta_m), \qquad \eta_m\in H$$

where $\bar{\Phi}$ is any isomorphism $\tilde{\Phi}:\mathrm{Pol(S^2)[h]}\rightarrow \mathrm{U_h}/\mathcal{I}_\mathrm{h}$. A star product of this kind was first written down in Ref. [5], where $\tilde{\Phi}$ was chosen in terms of the Weyl map,

$$\tilde{\Phi}([\eta])=[W(\eta)],$$

and $c(h)=c$. We will denote this product by $\star_P$. It has the nice properties that it restricts to all the orbits in a neighborhood of the regular orbit and that it is "covariant",

$$gf_1\star gf_2=g(f_1\star f_2).$$

Nevertheless, it is not differential, as it was shown in [5].

Finally, it was proven in [2] that $U_\hbar/\mathcal{I}_\hbar$, with $c(\hbar)=l(l+\hbar)$ corresponds to the algebra of geometric quantization in the formalism of Ref. [6]

3. DIFFERENTIAL AND TANGENTIAL STAR PRODUCTS

In this section we want to consider differential star products on $\mathcal{G}^*$ and on Θ, and to see the relation with the algebraic approach of the previous section. In Ref. [7], the differential

deformations of a Poisson manifold X modulo gauge equivalence are shown to be in one to one correspondence with the formal Poisson structures

$$\alpha = h\alpha_1 + h^2\alpha_2 + \cdots , \qquad [\alpha, \alpha] = 0,$$

(α_i are bivector fields and $[\,\cdot\,,\,\cdot\,]$ is the Schouten-Nijenhuis bracket) modulo the action of formal paths in the diffeomorphism group. So for every Poisson structure β, one can associate canonically an equivalence class of star products, the one corresponding to $h\beta$. If there are formal structures starting with $h\beta$ which are not equivalent to $h\beta$ through a diffeomorphism path, then one has star products not equivalent to the canonical one such that

$$f \star g - g \star f = h\beta(f, g) \ \ \mathrm{mod(h)}.$$

In the case of symplectic manifolds, these structures are classified by $H^2(X)[[h]]$. Since the compact coadjoint orbits have non trivial second cohomology group, we have more than one equivalence class of differential star products with term of first order the same Poisson bracket.

In the case of $\mathcal{G}^*$ with the Kirillov Poisson structure, it depends on the Lie algebra cohomology of $\mathcal{G}$. So for a semisimple Lie algebra there is only one equivalence class [3]. $\star_S$ is a representative of this equivalence class. It is not tangential to the orbits, and in fact, it was shown in Ref. [8] that no tangential star product could be extended over 0 for a semisimple Lie algebra.

Nevertheless, a regular orbit has always a neighborhood that is regularly foliated, $\mathcal{N}_\Theta \approx \Theta \times \mathbb{R}^m$. Since the Poisson structure is tangential, the coordinates on $\mathbb{R}^m$ can be considered as parameters, so one has in fact a family of Poisson structures on Θ smoothly varying with the parameters p_i, $\beta_{p_1,\dots p_m}$. Kontsevich's construction of the canonical star product gives a star product smoothly varying with the parameters p_i, or, interpreting it in the other way, a tangential star product canonically associated to β. It follows that $\star_S$, when restricted to $\mathcal{N}_\Theta$, is equivalent to a tangential star product. We denote it by $\star_T$.

We have three different products,

$\star_S$. It is differential, not tangential and defined on $\mathcal{G}^*$.

$\star_P$. It is not differential, tangential and defined on $\mathcal{G}^*$. (The ideal has been chosen so that $c_i(h) = c_i^0$).

$\star_T$. It is differential, tangential and defined only on $\mathcal{N}_\Theta$.

$\star_S$ restricted to the polynomials is isomorphic to $\star_P$. There is then an algebra homomorphism

$$\varphi : (\mathrm{Pol}(\mathcal{G}^*)[[h]], \star_P) \to (C^\infty(\mathcal{G}^*)[[h]], *_S).$$

We have that

$$\varphi(p_i - c_i^0) = p_i - c_i^0,$$

so $\mathcal{I}_0 \subset \mathrm{Pol}(\mathcal{G}^*)[[h]]$ is sent by φ into $\mathcal{I}_0 \subset C^\infty(\mathcal{G}^*)[[h]]$.

Restricting $\star_S$ to $\mathcal{N}_\Theta$ we have an algebra homomorphism

$$\rho : (C^\infty(\mathcal{N}_\Theta)[[h]], \star_S) \to (C^\infty(\mathbb{N})[[h]], *_T).$$

It is not difficult to see that although in general ρ doesn't send polynomials into polynomials, the algebra homomorphism structure implies that [3]

$$\rho(p_i - c_i^0) = p_i - c_i^0.$$

By composing $\rho \circ \varphi$, one obtains an homomorphism from a non differential star product to a differential one, such that both star products are tangential and the ideal $\mathcal{I}_0$ is mapped into the ideal $\mathcal{I}_0$. The homomorphism passes to the quotient, so the algebraic star product described in Section 2 is shown to be homomorphic to the differentiable star product associated by Kontsevich's map.

We note that we have chosen an algebraic star product with $c_i(h) = c_i$. This star product is not the one obtained from geometric quantization. The differential approach to quantization and geometric quantization, although they have similar features in the case of $\mathbb{R}^{2n}$ [9], seem not to give for compact coadjoint orbits, homomorphic algebras.

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